

# Improved Condensation Methods for Eigenvalue Problems

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*The conventional procedure used to condense the solution of eigenvalue problems for recovery of the lowest modes is tested by application to practical example structures. Evaluations are made of eigenvalue accuracy with respect to numbers of retained solution vectors. It is shown that solutions are likely to be inaccurate except in the special case of when prior knowledge of the mode shapes is available. One improvement for recovering the lowest modes is to supplement the retained vectors with static loading displacement functions. A further remedy is to perform iterative repetitions of the solution procedure. Great improvements in accuracy can be achieved with only a few iterative cycles. These improvements are effective in the typical case of when only a few valid lowest-mode solutions are required and the order of the problem is large so that it becomes important to minimize the computational time by means of solution condensation.*

## I. Conventional Condensation Methods

Procedures to condense the order of the eigenvalue problem solution for structural analysis are frequently used for computational efficiency and economy. Conventional condensation procedures are described in Refs. 1, 2, and 3.

Although specific descriptions and details of implementations vary, these approaches to condensation could be considered as Rayleigh-Ritz approximation techniques (Ref. 4). The objective is to solve the following homogeneous system of  $N$  equations for the unknown displacements  $\{X\}$ :

$$[M] \{\ddot{X}\} + [K] \{X\} = \{0\} \quad (1)$$

where  $M$  is the mass matrix and  $K$  is the stiffness matrix. The solution consists of the eigenvectors and the corresponding eigenvalues, which are measures of the modal natural frequencies.

A reduction in the order of the system from  $N$  to  $L$  ( $L < N$ ) can be effected by the transformation

$$\{X\} = [\gamma] \{\xi\} \quad (2)$$

where  $[\gamma]$  is an assumed transformation matrix of order  $N \times L$  and  $\{\xi\}$  is a set of generalized coordinates associated with the transformation. This reduces the order of the problem to  $L$ , and the equations to be solved become

$$[M] \{\ddot{\xi}\} + [K] \{\xi\} = 0 \quad (3)$$

where

$$[\underline{M}] = [\gamma]^t [M] [\gamma] \quad (3a)$$

and

$$[\underline{K}] = [\gamma]^t [K] [\gamma] \quad (3b)$$

The solution to Eq. (3) produces the eigenvalues and the modal matrix of eigenvectors  $[\phi]$  associated with the set of generalized coordinates. The set of  $L$  eigenvectors of order  $N$  for the original system can be recovered from

$$[X] = [\gamma] [\phi] \quad (4)$$

The transformation matrix, which is the key to the procedure, can be considered as a set of  $L$   $N$ -component vectors, with each vector equivalent to an assumed displacement function. The approximation of the solution depends upon how closely the assumed displacement functions (or linear combinations of them) agree with the actual eigenvectors.

A simple way to generate a set of displacement functions for the transformation matrix is to make a selection of a set of  $L$  "indicator" degrees of freedom, and then to develop static displacement vectors consistent with either unit displacements (Ref. 1) or unit loads (Refs. 2 and 3) at each indicator. Whether displacements or loads should be used to generate the displacement functions is theoretically immaterial; the logical choice depends upon convenience of execution for the available software. Consequently, the major requirement in proceeding with a condensed solution is the selection of suitable indicator degrees of freedom. In the case of regular classical structures, such as beams or strings, the true mode shapes are well known and the choice is not difficult nor critical. However, for practical structures of irregular geometry and properties, the important degrees of freedom are not clearly identifiable. Therefore, success can depend upon chance.

## II. Accuracy Tests of Conventional Method

The accuracy of condensed solutions was assessed for the analytical models of two structures used in the field of radar antennas. These particular models are of interest because they represent practical illustrations of irregular structures, and depart considerably from classical examples.

The first model is of an experimental structure built to simulate a 30-deg sector of a 26-m-diameter reflector. Its framing is shown in Fig. 1. The structure mass is about

2400 kg and consists of structural steel tees and angles. The reflecting surface comprises an additional 360 kg of nonstructural panels supported on the top chords of the trusses. Joint connections were idealized as pinned, with the exception of a few cases in which relatively stiff members were continuous across the joints. The mass matrix for the full model represents 190 translational degrees of freedom. The order of the stiffness matrix is slightly larger because of a few rotational degrees of freedom.

The second model is of the pedestal that provides the support and driving mechanism for the polar wheel of a 26-m-diameter reflector. Figure 2 shows a view of the framework. Bearings on the polar shaft support the polar wheel; the polar drive is by pinions at the drive skid. The mass of the framing members—steel angles, tees, pipes, and wide-flange beams—is about 42,000 kg. Additional concentrations, which total 18,000 kg, are distributed to joints in the vicinity of the drive skid and at the upper and lower bearings of the polar shaft. The mass matrix for the full model has 165 translational degrees of freedom. The stiffness matrix, with rotations included, contains 195 degrees of freedom.

The orders of both of these models were so small that it was feasible to solve the eigenvalue problems without condensation. After this, the solution accuracy for the first four frequencies and mode shapes was tracked through successive condensation solutions with diminishing numbers of indicator degrees of freedom. These indicators were originally chosen as carefully as would be expected in ordinary practice.

Figures 3 and 4 show the errors in frequency plotted against the percent of degrees of freedom retained. This percentage is for the ratio of the number of indicator degrees of freedom to the order of the uncondensed mass matrix. Comparisons of mode shapes were also made for the cases represented by examining the eigenvector components with the largest magnitudes. It was found, in general, that the eigenvector errors were considerably larger than the frequency errors. In some instances, the mode shapes from condensed solutions with small frequency errors were found to be unrelated to the mode shapes from the full solution. Usually, if frequencies were in error by more than about 3 to 5%, the mode shapes were no better than poor; frequency errors greater than 10 to 15% were accompanied by mode shapes that were rated from very poor to completely unrelated to the full-solution mode shapes.

According to a conventional rule of thumb, the ratio of the number of degrees of freedom in the model to the

number of accurate modes in the solution is between two and three. Therefore, according to this rule, the first four modes should be reproduced for either of these two models by retaining less than 10% of the full number of degrees of freedom. From another standpoint, it would not seem reasonable to consider condensation procedures to be highly effective unless accurate solutions for several modes could be obtained consistently with at least an order of magnitude reduction in the solution size. With these anticipations in mind, the results presented in Figs. 3 and 4, which show large errors for retentions greater than 10%, were highly disappointing.

However, the procedure was not abandoned at this point; instead, an entirely different approach was used for selection of the indicator degrees of freedom. This was an after-the-fact selection, in which the mode shapes from the uncondensed solutions were inspected to select the degrees of freedom that actually had the largest motions. In an attempt to identify and retain certain key degrees of freedom that apparently controlled solution validity, trial-and-error changes were also made in the retained set of indicators as the number of indicators in the set was being reduced.

The results of the modified selection of indicators, shown in Figs. 5 and 6, are a substantial improvement. For both models, the first four modes have been recovered with the order of the condensed eigenvalue problem solution (as given by the number of indicators) equal to less than 10% of the order of the uncondensed mass matrix.

### III. Evaluation of Results of Conventional Method

The success achieved for the modified selection confirms the theoretical validity of the conventional method but is of little practical help to the analyst. Although it was found that a selection of indicator degrees of freedom that included the degrees of freedom with the largest modal displacements tended to produce a substantial improvement in the validity of the solution, to make such a selection would require either clairvoyance or prior knowledge. It was also found neither necessary nor sufficient to choose degrees of freedom with the largest entries in the mass matrix, which is often used as a guideline for selection.

Another unfortunate property of the conventional method is the failure to converge toward improved accuracy. Addition of indicators to a poorly chosen set would sometimes closely reproduce the current inaccurate solution. The danger here is that the consistency of the results

could be misinterpreted as evidence of accuracy. Accuracy appears to depend upon the choice of a set of indicators (not necessarily a large set) that contains key, but elusive degrees of freedom. In some instances, the omission or addition of a single key indicator triggered the loss or recovery of a complete mode.

One more difficulty of the conventional method is that success or failure may be model-dependent. Better success was observed here in original selections for the reflector model than for the pedestal model. One reason for the difference may be the separation of the natural modes. It seems that the best results are obtained for analytical models in which the modes have the widest frequency separation.

### IV. Modifications for Conventional Method

Two modifications to the conventional method are effective in increasing reliability and removing some of the dependence upon the analyst's intuition or prior knowledge. The first consists of augmenting the indicator-derived displacement functions with up to six additional "static loading" displacement vectors. The second involves improving accuracy by iterative recycling of the Rayleigh-Ritz procedure.

Iterative solutions within a reduced space have been proposed previously. Jennings and Orr (Ref. 5) described a simultaneous iteration method that employs an orthogonal set of trial vectors, and they also indicate an extension that can be used for solutions of unconstrained structures. Ojalvo and Newman (Ref. 6) proposed a reduced-space solution employing a recursion algorithm to generate sequences of trial vectors. Whetstone and Jones (Ref. 7) used an initial Rayleigh-Ritz method based upon a selection of static force and rigid-body displacement vectors as the starting point for a reduced-space Stodola method solution that generates orthogonal eigenvectors one at a time. More recently, Dong, Wolf, and Peterson<sup>1</sup> employed iterative repetitions of the Rayleigh-Ritz method. The iterative Rayleigh-Ritz method, which is possibly the simplest of all to implement within existing analysis systems, is examined here with respect to accuracy and economy for practical applications. Additional displacement functions for the first modification can be developed within the context of a standard matrix interpretive analysis formulation such as the Structural Analysis and Matrix Interpretive System (SAMIS) program. The recycling modification, which also appears to have more potential,

<sup>1</sup>In an article to be published in the *Int. J. Numer. Meth. Eng.*

has been readily incorporated within both the SAMIS and NASA Structural Analysis (NASTRAN) programs.

#### A. Augmented Displacement Function Modification

The additional displacement functions are developed by first determining the six rigid-body displacements of the structure for independent unit motions (three translations and three rotations) of the foundation. A loading matrix is derived by post-multiplying the mass matrix by the  $N \times 6$  matrix of rigid-body displacements. That is, let  $[\rho]$  be the set of rigid-body displacements and let  $[P]$  be the loading matrix; then

$$[P] = [M] [\rho] \quad (5)$$

This loading is applied to the structure, and the corresponding displacements are added to the displacements constructed from the indicators. The resulting set of displacement functions becomes the transformation used in the condensed solution. Thus, let  $[\gamma_R]$  be the additional static-loading displacement functions, and let  $[\gamma_I]$  be the displacement functions generated by indicators; then  $[\gamma_R]$  is found from the solution of

$$[K] [\gamma_R] = [P] \quad (6)$$

and

$$[\gamma] = [\gamma_I, \gamma_R] \quad (7)$$

From this point on, the solution proceeds in the conventional way (see Eq. 2). The order of the condensed eigenvalue problem is now  $L + 6$ . If  $L$  is small, the increase can be readily accommodated in the solution; if  $L$  is large, the increase is relatively small. Therefore, the increase in the size of the solution and the computation time caused by the additional functions is not significant. In a limited number of evaluations of the relative accuracy improvement, it appears that the rigid-body translations have a larger influence than the rigid-body foundation rotations. Hence, the effort in generating program input to define foundation rotations may not be worthwhile.

The added displacement functions constructed from the rigid-body translations are proportional to the displacements that result from loading the structure with its own weight applied sequentially in the directions of the foundation motions. In fact, the suggested additional displacement functions were motivated by typically successful applications of Rayleigh's method in recovering first-mode frequencies with the assumption of a single static self-weight loading displacement function. Therefore, the

modification will ordinarily ensure the recovery of the fundamental frequency, and thereby remove at least one defect that has occasionally been noted in the conventional method.

#### B. Recycling Modification

The recycling modification is applied after an initial solution using indicator functions (optionally augmented by the static loading functions). The initial mode shapes for the full  $N$  degrees of freedom are recovered as in Eq. (4) and are then applied as in Eq. (5) (in place of the rigid-body displacements) to generate new loading vectors, e.g.,

$$[X^0] = [\gamma^0] [\phi^0] \quad (8)$$

$$[P^1] = [M] [X^0] \quad (9)$$

After this, the transformation matrix for the first additional cycle is constructed by solving

$$[K] [\gamma^1] = [P^1] \quad (10)$$

The eigenvalue problem solution is then regenerated according to Eq. (3). This procedure can be repeated iteratively until convergence to within a specified criterion is achieved. Equations (8), (9), and (10) are applicable with incremented superscripts at each subsequent cycle. A mathematical proof of eventual cyclic convergence need not be supplied; it is only necessary to consider this as an extension of Stodola's procedure of matrix iteration (Ref. 4), for which convergence is readily established (Ref. 8). That is, instead of iterating on one trial modal vector at a time, the present modification is an extension to a block iteration that operates upon several vectors simultaneously. Here, the eigenvalue problem is solved repeatedly in a greatly reduced space, which is in contrast to the classical uncondensed solution procedures that perform only one solution of the problem in the full space of the original mass matrix. Therefore, to achieve solution economy, the emphasis is now placed upon efficient performance of the operations leading to the formation of Eq. (3) rather than upon the procedures used in obtaining solutions to this equation.

## V. Results and Discussion of Modifications

The effects of the modifications to the procedures for the pedestal model are shown in Fig. 7. Three curves are included for each mode: curve A is for reference—to show the accuracy for the conventional initial solution with

indicator functions only; curve B shows the accuracy for the first modification—that is, six static-loading functions have been added to the indicator functions of curve A; and curve C represents the accuracy obtained with the addition of only one new iteration cycle and starting with the mode shapes associated with the points of curve B. It can be seen in comparison with the conventional solution that solutions with the six new displacement functions are often several times more accurate, and that solutions from the iteration cycle are typically more accurate by more than one order of magnitude.

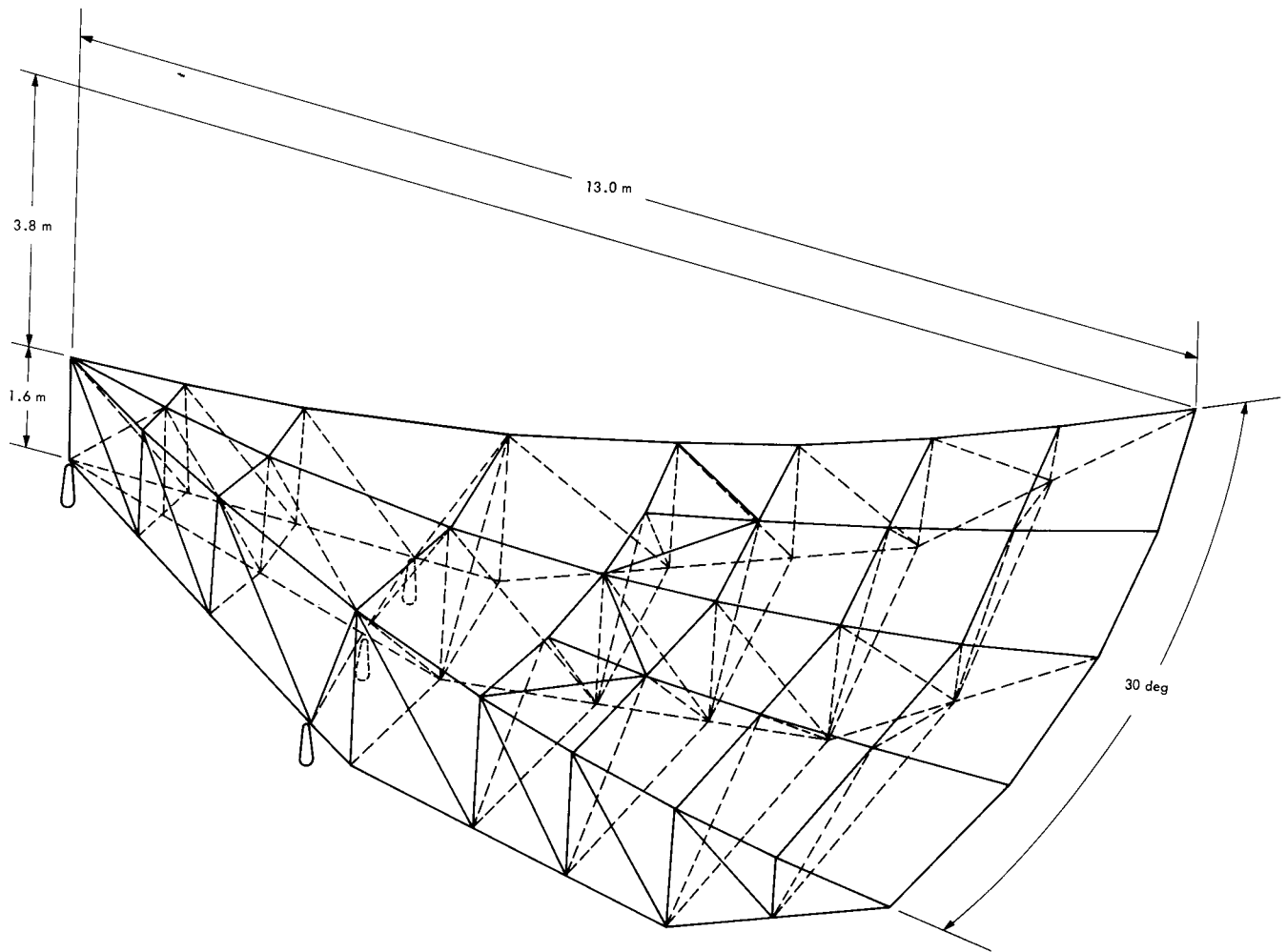
Most of the indicators selected for the curves of Fig. 7 were from the after-the-fact selection, and consisted of a relatively small set. In a few cases, the addition of one iteration to the initial solution for larger sets of indicators produced excellent accuracy for relatively larger numbers of natural modes. In some of these examples, the particular set of indicators had produced invalid results on the initial cycle. For a larger system a solution was obtained without condensation for an analytical model of an antenna reflector that contained about 1300 degrees of freedom. Only the first modal solution was generated, which required about 32 min of computation time (Univac 1108-Exec 8 computer) in the eigenvalue solution phase. After

this, a Guyan reduction solution was performed with 29 indicator degrees of freedom and continued through five additional iteration cycles. The initial solution was completed after 15 min, the first iteration required 13 min, and each of the four subsequent iteration cycles took 8 min. The changes in modal frequencies at each cycle diminish approximately according to a geometric progression. This relationship was used to extrapolate the modal frequencies at convergence. Figure 8a shows the numbers of natural frequencies obtained cyclically with respect to limiting percentages of variation from the extrapolated convergent frequencies. Figure 8b indicates the estimated computer time saved if the same number of modes that were obtained with less than 0.05% variation from the convergent frequencies had been derived by the uncondensed model.

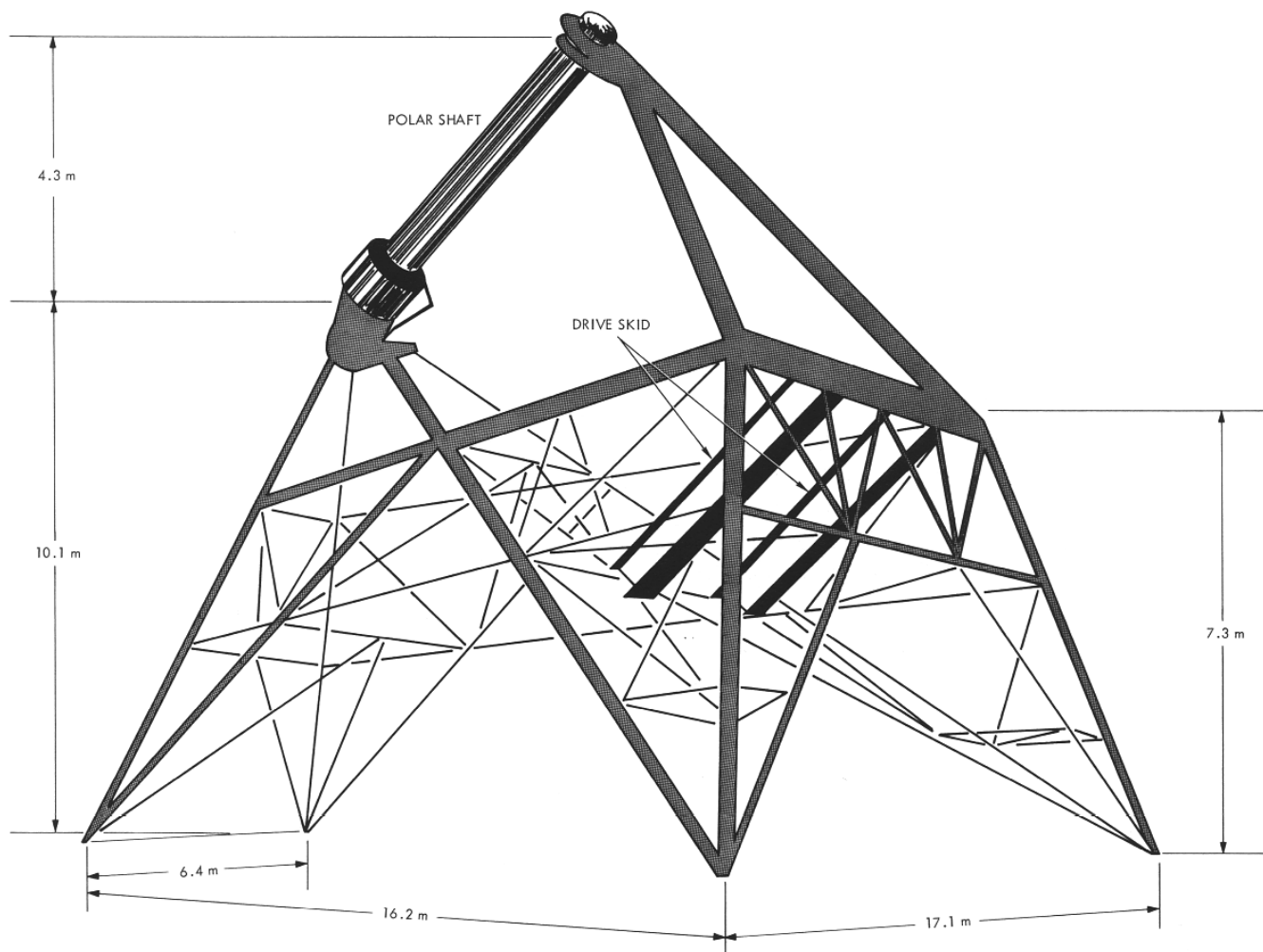
Both types of the modifications have been implemented within the JPL SAMIS program by means of a few pseudo-instructions. The recycling modification has been implemented in a JPL-parochial version of the NASTRAN program, called "Level 14.0.2." This program level permits the specification of convergence criteria on natural frequency or generalized mass and also provides the option of truncating the number of solution vectors that are retained after a specified number of iteration cycles.

## References

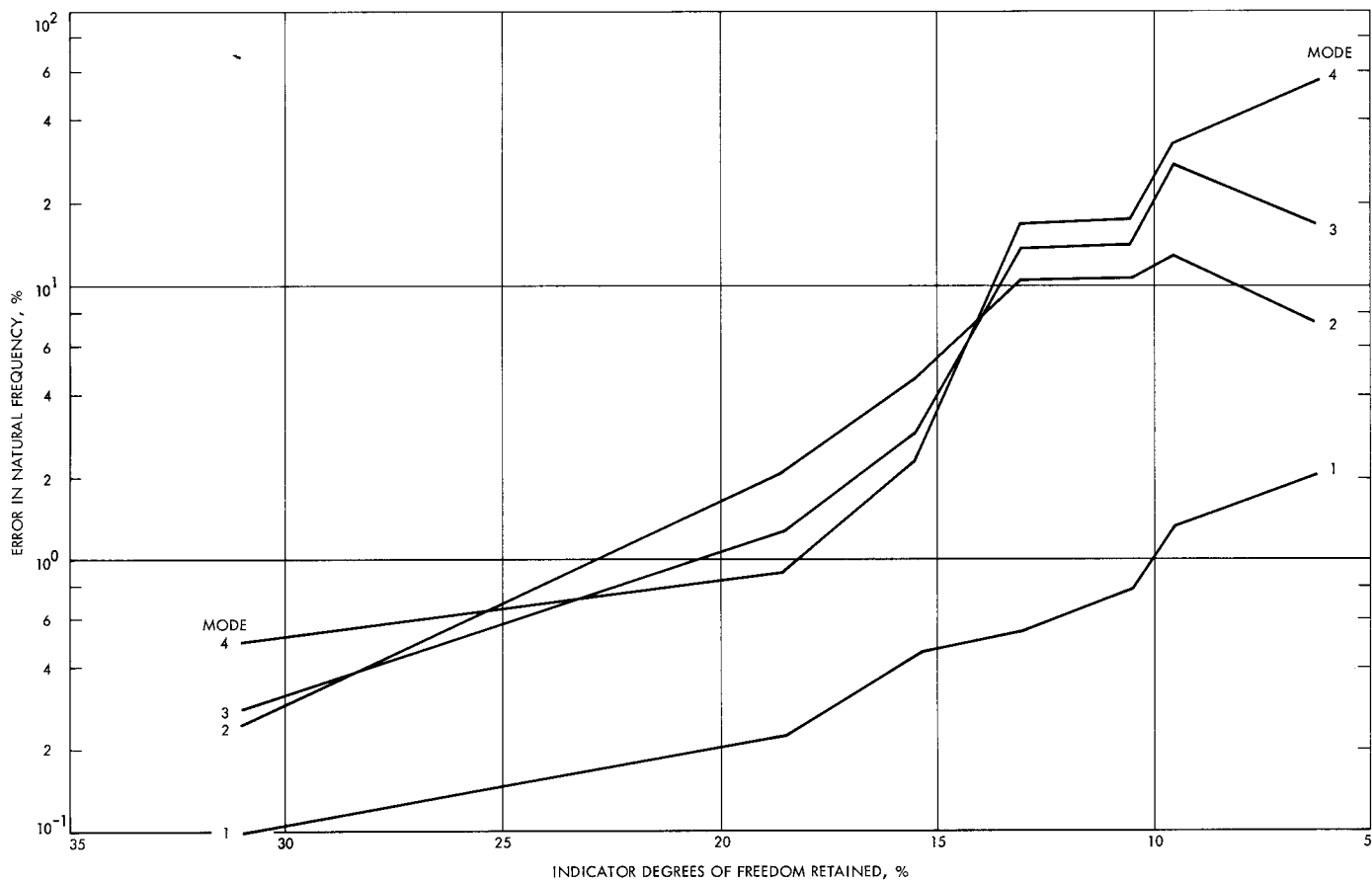
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**Fig. 1. Thirty-degree reflector sector structure**

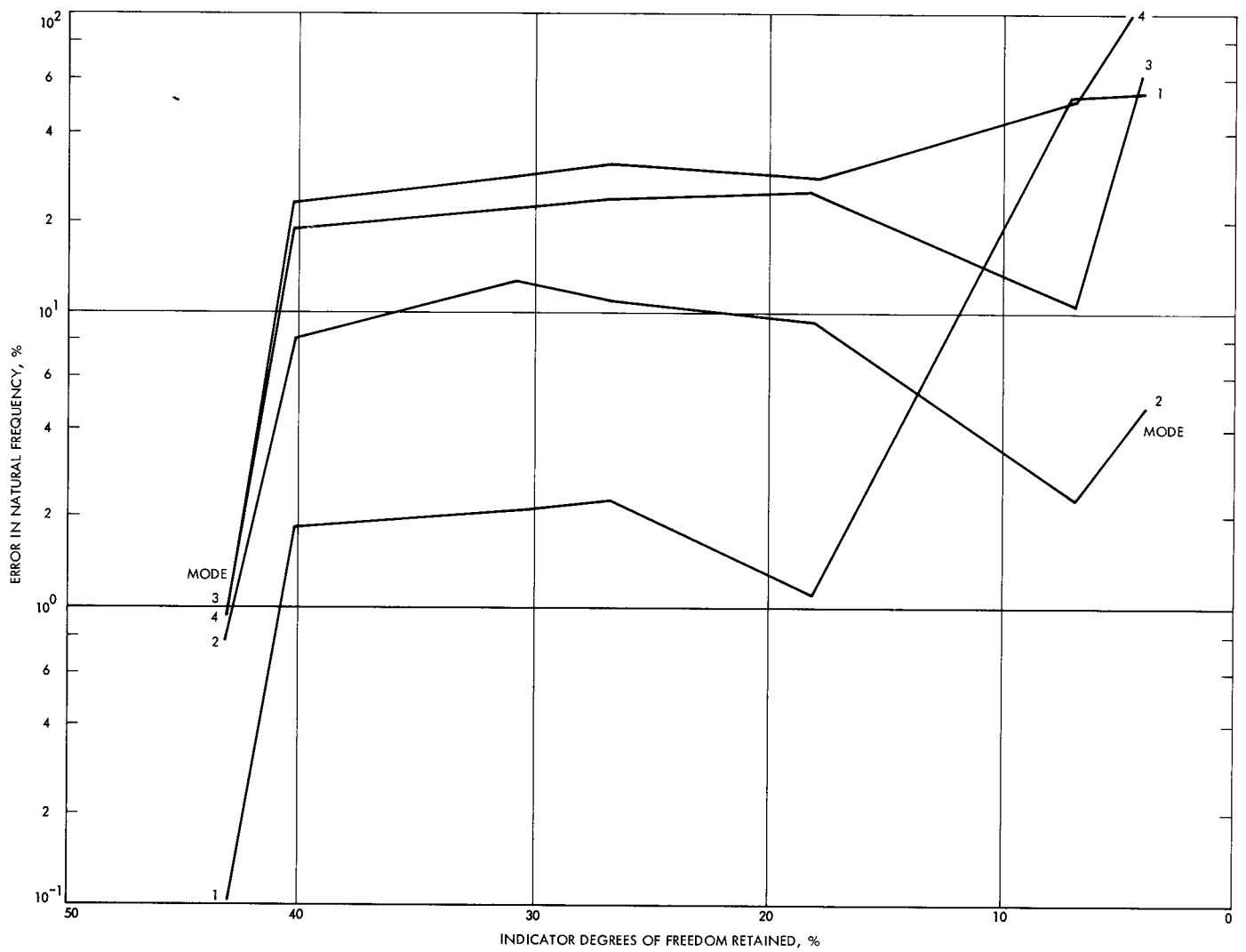


**Fig. 2. Pedestal structure**

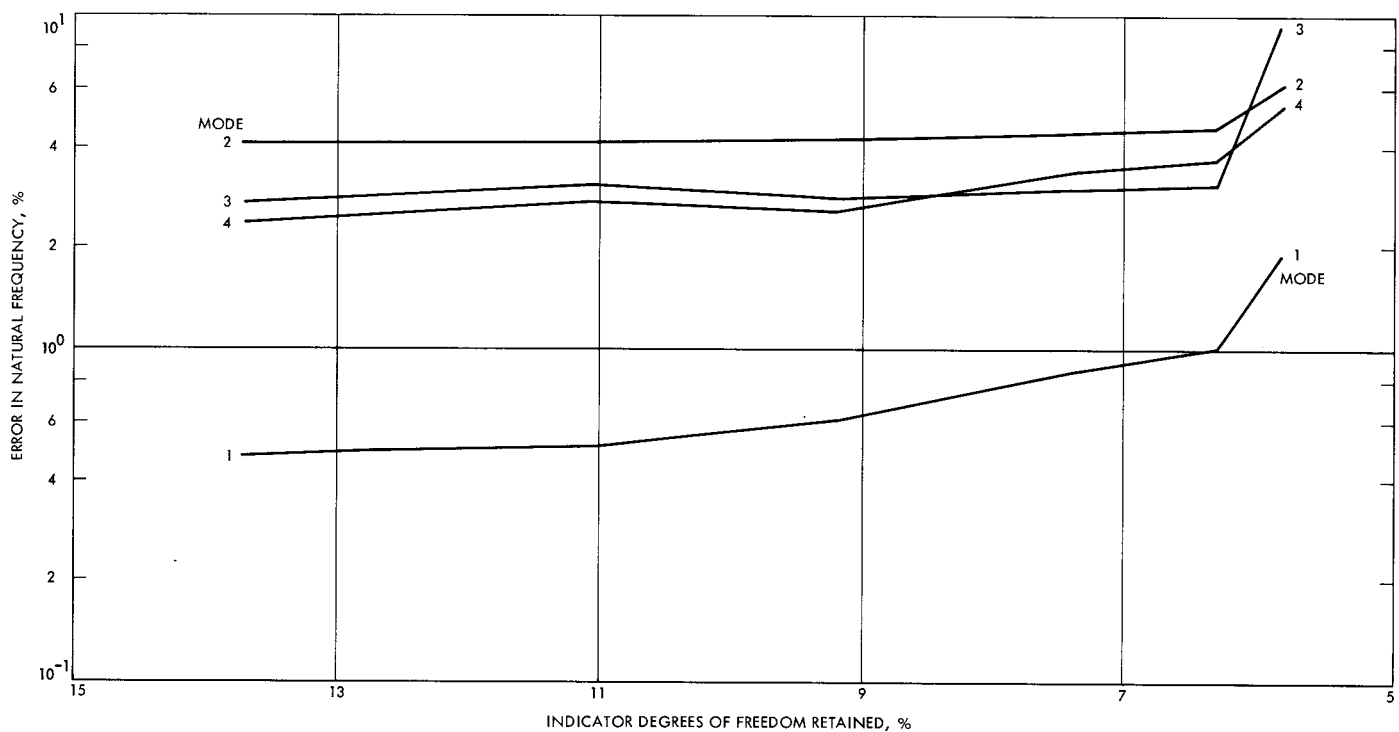


**Fig. 3. Accuracy of condensed solution for reflector sector structure: original indicator selections**

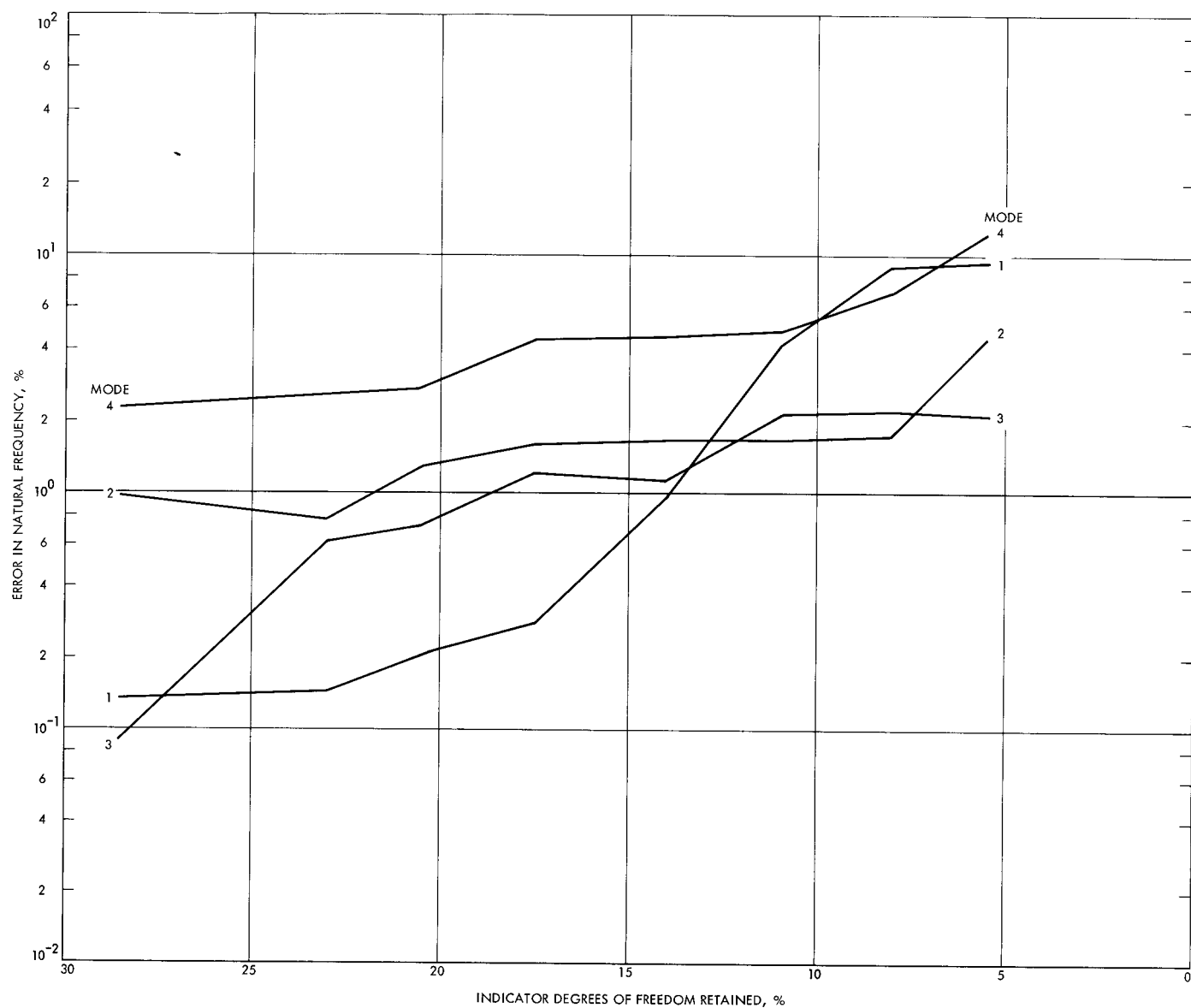




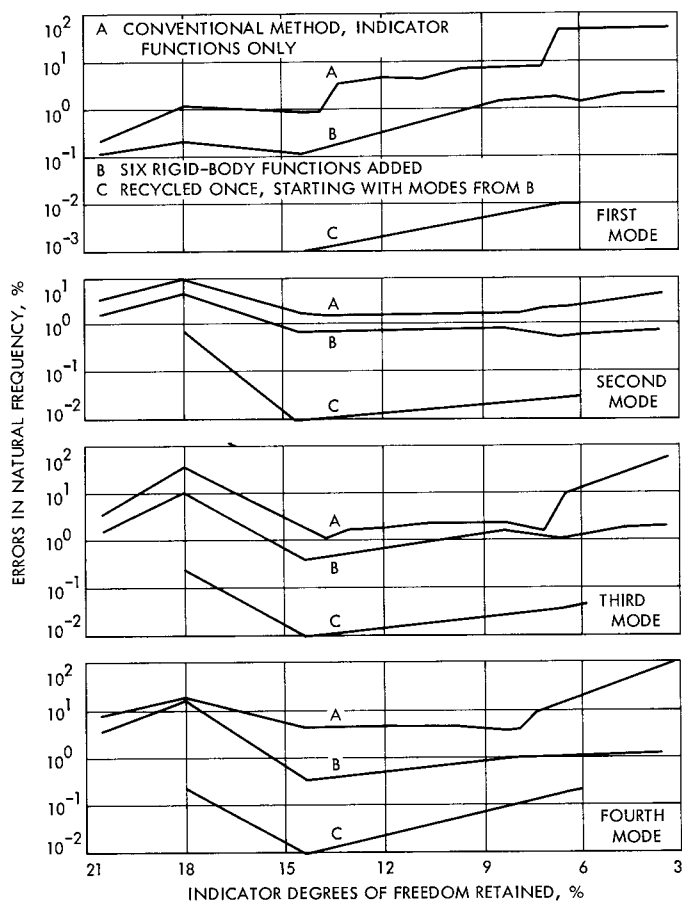
**Fig. 4. Accuracy of condensed solution for pedestal structure: original indicator selections**



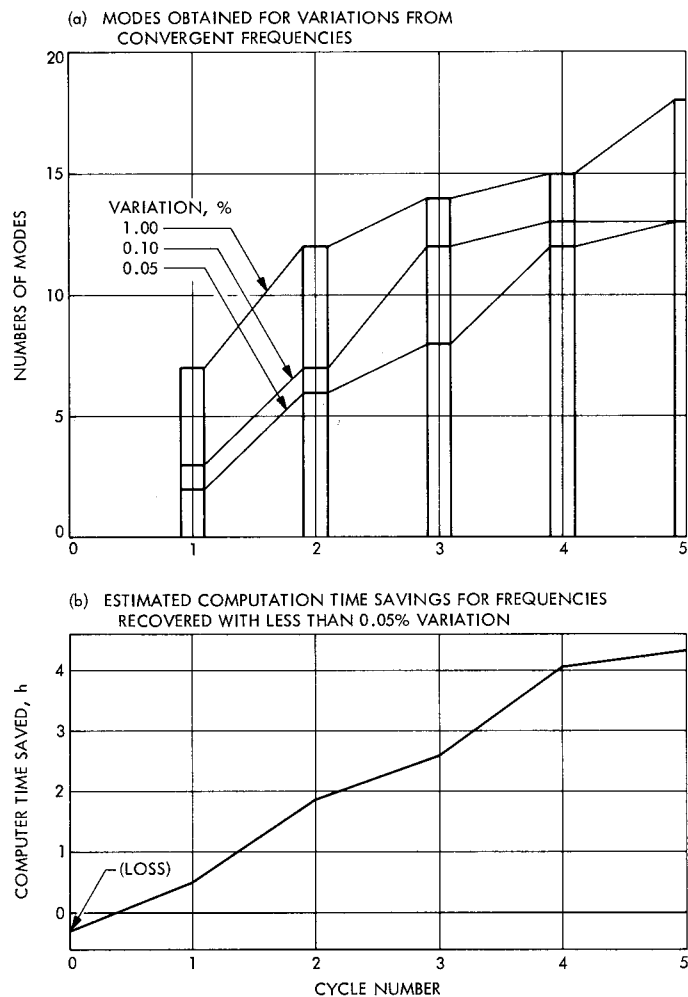
**Fig. 5. Accuracy of condensed solution for reflector sector structure: after-the-fact indicator selections**



**Fig. 6. Accuracy of condensed solution for pedestal structure: after-the-fact indicator selections**



**Fig. 7. Improvement of pedestal structure solution accuracy**



**Fig. 8. Recycled solutions for antenna reflector**